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Solutions of the system of maximum difference equations $x_{n+1}=\max\left\{\frac{1}{x_{n-3}},\frac{y_n}{x_n}\right\}; y_{n+1}=\max\left\{\frac{1}{y_{n-3}},\frac{x_n}{y_n}\right\}$

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ABSTRACT

The behaviour of the solutions of the following system of difference equations is examined,

$$x_{n+1}=\max\left\{\frac{1}{x_{n-3}},\frac{y_n}{x_n}\right\}; y_{n+1}=\max\left\{\frac{1}{y_{n-3}},\frac{x_n}{y_n}\right\},$$

where the initial conditions are positive real numbers.

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1. Introduction

Latterly, there has been a great concern in studying the periodic nature of nonlinear difference equations. Although difference equations are relatively simple in form, it is, unfortunately, extremely diffucult to understand thorougly the periodic behavior of their solutions. The periodic nature of nonlinear difference equations of the max type has been investigated by many authors. See for example [1-34].

In this paper, we investigated of the solutions of the following system of diffference equations

$$x_{n+1}=\max\left\{\frac{1}{x_{n-3}},\frac{y_n}{x_n}\right\}; y_{n+1}=\max\left\{\frac{1}{y_{n-3}},\frac{x_n}{y_n}\right\} \quad (1)$$

where the initial conditions are positive real numbers.

Definition 1: Let I be an interval of real numbers and let $f: I^{s+1}=I$ be a continuously differentiable function where s is a non-negative integer. Consider the difference equation

$$x_{n+1}=f\left(x_n,x_{n-1},...,x_{n-s}\right) \quad (2)$$

with the initial values $x_{-s},...,x_0 \in I$. A point $\overline{x}$ called an equilibrium point of equation.(2) if $\overline{x} = f\left(\overline{x},...,\overline{x}\right)$.

Definition 2: A positive semicycle of a solutions $\{x_n\}_n^{\infty} = -s$ of equation (2) consist of a string of terms $\{x_l, x_{l+1},...,x_m\}$ all greater than or equal to equilibrium $\overline{x}$ with $l \geq -s$ and $m \leq \infty$ such that either $l = -s$ or $l > -s$ and $x_{l-1} < \overline{x}$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} < \overline{x}$.

Definition 3: A negative semicycle of a solutions $\{x_n\}_n^{\infty} = -s$ of equation (2) consist of a string of terms $\{x_l, x_{l+1},...,x_m\}$ all less than or equal to equilibrium $\overline{x}$ with $l \geq -s$ and $m \leq \infty$ such that either $l = -s$ or $l > -s$ and $x_{l-1} \geq \overline{x}$ and either $m = \infty$ or $m \leq \infty$ and $x_{m+1} \geq \overline{x}$.

2. Main results

Let $\overline{x}$ and $\overline{y}$ be the unique positive equilibrium of equation. (1), then clearly,

$$\overline{x} = \max\left\{\frac{1}{\overline{x}}, \frac{\overline{y}}{\overline{x}}\right\}; \overline{y} = \max\left\{\frac{1}{\overline{y}}, \frac{\overline{x}}{\overline{y}}\right\}$$

$$\overline{x} = \frac{1}{\overline{x}} \Rightarrow \overline{x}^2 = 1 \Rightarrow \overline{x} = 1,$$

$$\overline{y} = \frac{1}{\overline{y}} \Rightarrow \overline{y}^2 = 1 \Rightarrow \overline{y} = 1,$$

and, we can obtain $\overline{x} = 1$ and $\overline{y} = 1$.

Lemma 1: Assume that,

$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-3} < y_{-2} < y_{-1},\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-3} < y_{-1} < y_{-2},$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-2} < y_{-3} < y_{-1},\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-2} < y_{-1} < y_{-3},$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-1} < y_{-2} < y_{-3},\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-1} < y_{-3} < y_{-2}.$$

Then the following statements are true for the solutions of equation (1):

a) Every positive semicycle of length one is followed by a negative semicycle of length one for the first four terms.
b) Every negative semicycle of length one is followed by a positive semicycle of length one for the first four terms.
c) The last four terms of the solutions are positive.

Proof: Let consider the following initial conditions.

$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-3} < y_{-2} < y_{-1},\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-3} < y_{-1} < y_{-2},$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-2} < y_{-3} < y_{-1},\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-2} < y_{-1} < y_{-3},$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-1} < y_{-2} < y_{-3},\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-1} < y_{-3} < y_{-2}.$$

The solution x_n and y_n can be obtained as follows:

$$x_1 = \max\left\{\frac{1}{x_{-3}}, \frac{y_0}{x_0}\right\} = \frac{y_0}{x_0} > \overline{x};\ \ y_1 = \max\left\{\frac{1}{y_{-3}}, \frac{x_0}{y_0}\right\} = \frac{x_0}{y_0} < \overline{y};$$

$$x_2 = \max\left\{\frac{1}{x_{-2}}, \frac{y_1}{x_1}\right\} = \max\left\{\frac{1}{x_{-2}}, \frac{x_0^2}{y_0^2}\right\} = \frac{x_0^2}{y_0^2} < \overline{x};\ \ y_2 = \max\left\{\frac{1}{y_{-2}}, \frac{x_1}{y_1}\right\} = \max\left\{\frac{1}{y_{-2}}, \frac{y_0^2}{x_0^2}\right\} = \frac{y_0^2}{x_0^2} > \overline{y};$$

$$x_3=\max\left\{\frac{1}{x_{-1}},\frac{y_2}{x_2}\right\}=\max\left\{\frac{1}{x_{-1}},\frac{y_0^4}{x_0^4}\right\}=\frac{y_0^4}{x_0^4}>\bar{x};\quad y_3=\max\left\{\frac{1}{y_{-1}},\frac{x_2}{y_2}\right\}=\max\left\{\frac{1}{y_{-1}},\frac{x_0^4}{y_0^4}\right\}=\frac{x_0^4}{y_0^4}<\bar{y};$$

$$x_4=\max\left\{\frac{1}{x_0},\frac{y_3}{x_3}\right\}=\max\left\{\frac{1}{x_0},\frac{x_0^8}{y_0^8}\right\}=\frac{x_0^8}{y_0^8}<\bar{x};\quad y_4=\max\left\{\frac{1}{y_0},\frac{x_3}{y_3}\right\}=\max\left\{\frac{1}{y_0},\frac{y_0^8}{x_0^8}\right\}=\frac{y_0^8}{x_0^8}>\bar{y};$$

$$x_5=\max\left\{\frac{1}{x_1},\frac{y_4}{x_4}\right\}=\max\left\{\frac{x_0}{y_0},\frac{y_0^4}{x_0^4}\right\}=\frac{y_0^{16}}{x_0^{16}}>\bar{x};\quad y_5=\max\left\{\frac{1}{y_{-1}},\frac{x_2}{y_2}\right\}=\max\left\{\frac{y_0}{x_0},\frac{x_0^7}{y_0^8}\right\}=\frac{y_0}{x_0}>\bar{y};$$

$$x_6=\max\left\{\frac{1}{x_2},\frac{y_5}{x_5}\right\}=\max\left\{\frac{y_0^2}{x_0^2},\frac{x_0^6}{y_0^7}\right\}=\frac{y_0^2}{x_0^2}>\bar{x};\quad y_6=\max\left\{\frac{1}{y_2},\frac{x_5}{y_5}\right\}=\max\left\{\frac{x_0^2}{y_0^2},\frac{y_0^{15}}{x_0^{15}}\right\}=\frac{y_0^{15}}{x_0^{15}}>\bar{y};$$

$$x_7=\max\left\{\frac{1}{x_3},\frac{y_6}{x_6}\right\}=\max\left\{\frac{x_0^4}{y_0^4},\frac{y_0^{13}}{x_0^{13}}\right\}=\frac{y_0^{13}}{x_0^{13}}>\bar{x};\quad y_7=\max\left\{\frac{1}{y_3},\frac{x_6}{y_6}\right\}=\max\left\{\frac{y_0^4}{x_0^4},\frac{x_0^{13}}{y_0^{13}}\right\}=\frac{y_0^4}{x_0^4}>\bar{y};$$

$$x_8=\max\left\{\frac{1}{x_4},\frac{y_7}{x_7}\right\}=\max\left\{\frac{y_0^8}{x_0^8},\frac{x_0^9}{y_0^9}\right\}=\frac{y_0^8}{x_0^8}>\bar{x};\quad y_8=\max\left\{\frac{1}{y_4},\frac{x_7}{y_7}\right\}=\max\left\{\frac{x_0^8}{y_0^8},\frac{y_0^9}{x_0^9}\right\}=\frac{y_0^9}{x_0^9}>\bar{y};$$

$$\vdots$$

Hence, we obtained:

$$x_1>\bar{x},\ x_2<\bar{x},\ x_3>\bar{x},\ x_4<\bar{x},\ x_5>\bar{x},\ x_6>\bar{x},\ x_7>\bar{x},\ x_8>\bar{x},\ x_9>\bar{x},\dots$$
$$y_1<\bar{y},\ y_2>\bar{y},\ y_3<\bar{y},\ y_4>\bar{y},\ y_5>\bar{y},\ y_6>\bar{y},\ y_7>\bar{y},\ y_8>\bar{y},\ y_9<\bar{y},\dots\ .$$

Hence, the solution x_n, y_n for $n\geq 0$, every positive semicycle consists of one terms for first four terms, every negative semicycle consists of one terms for first four terms. Hence, last four terms of the solutions are positive.
The following Lemmas 2, 3 and 4 can be obtained similarly to Lemma 1.

Lemma 2: Assume that,

$$1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-3}<y_{-2}<y_{-1}<y_0,\ 1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-3}<y_{-1}<y_{-2}<y_0,$$
$$1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-2}<y_{-3}<y_{-1}<y_0,\ 1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-2}<y_{-1}<y_{-3}<y_0,$$
$$1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-1}<y_{-3}<y_{-2}<y_0,\ 1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-1}<y_{-2}<y_{-3}<y_0,$$
$$1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-3}<y_{-2}<y_0<y_{-1},\ 1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-3}<y_{-1}<y_0<y_{-2},$$
$$1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-2}<y_{-3}<y_0<y_{-1},\ 1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-2}<y_{-1}<y_0<y_{-3}.$$

Then the following statements are true for the solutions of equation (1):

a) Every positive semicycle of length one is followed by a negative semicycle of length one for the first four terms.
b) Every negative semicycle of length one is followed by a positive semicycle of length one for the first four terms.
c) The last four terms of the solutions are positive.

Lemma 3: Assume that,

$$1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-3}<y_0<y_{-1}<y_{-2},\ 1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-3}<y_0<y_{-2}<y_{-1},$$
$$1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-2}<y_0<y_{-1}<y_{-3},\ 1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-2}<y_0<y_{-3}<y_{-1},$$
$$1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-1}<y_0<y_{-2}<y_{-3},\ 1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-1}<y_0<y_{-3}<y_{-2}.$$

Then the following statements are true for the solutions of equation (1) :

a) Every positive semicycle of length one is followed by a negative semicycle of length one for the first four terms.
b) Every negative semicycle of length one is followed by a positive semicycle of length one for the first four terms.
c) The last four terms of the solutions are positive.

Lemma 4: Assume that,

$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-2} < y_0 < y_{-3} \ , \ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-3} < y_0 < y_{-2} .$$

Then the following statements are true for the solutions of equation (1):

a) Every positive semicycle of length one is followed by a negative semicycle of length one for the first four terms.
b) Every negative semicycle of length one is followed by a positive semicycle of length one for the first four terms.
c) The last four terms of the solutions are positive.

Theorem 1: Let (x_n, y_n) be a solution of equation (1) for

$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-3} < y_{-2} < y_{-1} , \ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-3} < y_{-1} < y_{-2} ,$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-2} < y_{-3} < y_{-1} , \ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-2} < y_{-1} < y_{-3} ,$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-1} < y_{-2} < y_{-3} , \ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-1} < y_{-3} < y_{-2} .$$

Then for $n = 0,1,...$ we have:

$$x(n) = \left\{ \frac{y_0}{x_0}; \frac{x_0^2}{y_0^2}; \frac{y_0^4}{x_0^4}; \frac{x_0^8}{y_0^8}; \frac{y_0^{16}}{x_0^{16}}; \frac{y_0^2}{x_0^2}; \frac{y_0^{13}}{x_0^{13}}; \frac{y_0^8}{x_0^8}; \frac{y_0}{x_0}; \frac{x_0^2}{y_0^2}; \frac{y_0^4}{x_0^4}; \frac{x_0^8}{y_0^8}; \frac{y_0^{16}}{x_0^{16}}; \frac{y_0^2}{x_0^2}; \frac{y_0^{13}}{x_0^{13}}; \frac{y_0^8}{x_0^8}; ... \right\},$$
$$y(n) = \left\{ \frac{x_0}{y_0}; \frac{y_0^2}{x_0^2}; \frac{x_0^4}{y_0^4}; \frac{y_0^8}{x_0^8}; \frac{y_0}{x_0}; \frac{y_0^{15}}{x_0^{15}}; \frac{y_0^4}{x_0^4}; \frac{y_0^9}{x_0^9}; \frac{x_0}{y_0}; \frac{y_0^2}{x_0^2}; \frac{x_0^4}{y_0^4}; \frac{y_0^8}{x_0^8}; \frac{y_0}{x_0}; \frac{y_0^{15}}{x_0^{15}}; \frac{y_0^4}{x_0^4}; \frac{y_0^9}{x_0^9}; ... \right\}.$$

Proof: We obtain:

$$x_1 = \max\left\{ \frac{1}{x_{-3}}, \frac{y_0}{x_0} \right\} = \frac{y_0}{x_0}; \quad y_1 = \max\left\{ \frac{1}{y_{-3}}, \frac{x_0}{y_0} \right\} = \frac{x_0}{y_0};$$
$$x_2 = \max\left\{ \frac{1}{x_{-2}}, \frac{y_1}{x_1} \right\} = \max\left\{ \frac{1}{x_{-2}}, \frac{x_0^2}{y_0^2} \right\} = \frac{x_0^2}{y_0^2}; \quad y_2 = \max\left\{ \frac{1}{y_{-2}}, \frac{x_1}{y_1} \right\} = \max\left\{ \frac{1}{y_{-2}}, \frac{y_0^2}{x_0^2} \right\} = \frac{y_0^2}{x_0^2};$$
$$x_3 = \max\left\{ \frac{1}{x_{-1}}, \frac{y_2}{x_2} \right\} = \max\left\{ \frac{1}{x_{-1}}, \frac{y_0^4}{x_0^4} \right\} = \frac{y_0^4}{x_0^4}; \quad y_3 = \max\left\{ \frac{1}{y_{-1}}, \frac{x_2}{y_2} \right\} = \max\left\{ \frac{1}{y_{-1}}, \frac{x_0^4}{y_0^4} \right\} = \frac{x_0^4}{y_0^4};$$
$$x_4 = \max\left\{ \frac{1}{x_0}, \frac{y_3}{x_3} \right\} = \max\left\{ \frac{1}{x_0}, \frac{x_0^8}{y_0^8} \right\} = \frac{x_0^8}{y_0^8}; \quad y_4 = \max\left\{ \frac{1}{y_0}, \frac{x_3}{y_3} \right\} = \max\left\{ \frac{1}{y_0}, \frac{y_0^8}{x_0^8} \right\} = \frac{y_0^8}{x_0^8};$$
$$x_5 = \max\left\{ \frac{1}{x_1}, \frac{y_4}{x_4} \right\} = \max\left\{ \frac{x_0}{y_0}, \frac{y_0^4}{x_0^4} \right\} = \frac{y_0^{16}}{x_0^{16}}; \quad y_5 = \max\left\{ \frac{1}{y_{-1}}, \frac{x_2}{y_2} \right\} = \max\left\{ \frac{y_0}{x_0}, \frac{x_0^7}{y_0^8} \right\} = \frac{y_0}{x_0};$$
$$x_6 = \max\left\{ \frac{1}{x_2}, \frac{y_5}{x_5} \right\} = \max\left\{ \frac{y_0^2}{x_0^2}, \frac{x_0^6}{y_0^7} \right\} = \frac{y_0^2}{x_0^2}; \quad y_6 = \max\left\{ \frac{1}{y_2}, \frac{x_5}{y_5} \right\} = \max\left\{ \frac{x_0^2}{y_0^2}, \frac{y_0^{15}}{x_0^{15}} \right\} = \frac{y_0^{15}}{x_0^{15}};$$

$$x_7 = \max\left\{\frac{1}{x_3}, \frac{y_6}{x_6}\right\} = \max\left\{\frac{x_0^4}{y_0^4}, \frac{y_0^{13}}{x_0^{13}}\right\} = \frac{y_0^{13}}{x_0^{13}}; \quad y_7 = \max\left\{\frac{1}{y_3}, \frac{x_6}{y_6}\right\} = \max\left\{\frac{y_0^4}{x_0^4}, \frac{x_0^{13}}{y_0^{13}}\right\} = \frac{y_0^4}{x_0^4};$$

$$x_8 = \max\left\{\frac{1}{x_4}, \frac{y_7}{x_7}\right\} = \max\left\{\frac{y_0^8}{x_0^8}, \frac{x_0^9}{y_0^9}\right\} = \frac{y_0^8}{x_0^8}; \quad y_8 = \max\left\{\frac{1}{y_4}, \frac{x_7}{y_7}\right\} = \max\left\{\frac{x_0^8}{y_0^8}, \frac{y_0^9}{x_0^9}\right\} = \frac{y_0^9}{x_0^9};$$

$$\vdots$$

Thus,

$$x(n) = \left\{\frac{y_0}{x_0}; \frac{x_0^2}{y_0^2}; \frac{y_0^4}{x_0^4}; \frac{x_0^8}{y_0^8}; \frac{y_0^{16}}{x_0^{16}}; \frac{y_0^2}{x_0^2}; \frac{y_0^{13}}{x_0^{13}}; \frac{y_0^8}{x_0^8}; \frac{y_0}{x_0}; \frac{x_0^2}{y_0^2}; \frac{y_0^4}{x_0^4}; \frac{x_0^8}{y_0^8}; \frac{y_0^{16}}{x_0^{16}}; \frac{y_0^2}{x_0^2}; \frac{y_0^{13}}{x_0^{13}}; \frac{y_0^8}{x_0^8}; \ldots\right\},$$

$$y(n) = \left\{\frac{x_0}{y_0}; \frac{y_0^2}{x_0^2}; \frac{x_0^4}{y_0^4}; \frac{y_0^8}{x_0^8}; \frac{y_0}{x_0}; \frac{y_0^{15}}{x_0^{15}}; \frac{y_0^4}{x_0^4}; \frac{y_0^9}{x_0^9}; \frac{x_0}{y_0}; \frac{y_0^2}{x_0^2}; \frac{x_0^4}{y_0^4}; \frac{y_0^8}{x_0^8}; \frac{y_0}{x_0}; \frac{y_0^{15}}{x_0^{15}}; \frac{y_0^4}{x_0^4}; \frac{y_0^9}{x_0^9}; \ldots\right\}.$$

the solutions are shown to be 8-period.
The following Theorems 2, 3 and 4 can be obtained similarly to Theorem 1.

Theorem 2: Let (x_n, y_n) be a solution of equation (1) for

$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-2} < y_{-1} < y_0, \; 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-1} < y_{-2} < y_0,$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-3} < y_{-1} < y_0, \; 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-1} < y_{-3} < y_0,$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-3} < y_{-2} < y_0, \; 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-2} < y_{-3} < y_0,$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-2} < y_0 < y_{-1}, \; 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-1} < y_0 < y_{-2},$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-3} < y_0 < y_{-1}, \; 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-1} < y_0 < y_{-3}.$$

Then for $n = 0,1,\ldots$ we have:

$$x(n) = \left\{\frac{y_0}{x_0}; \frac{x_0^2}{y_0^2}; \frac{y_0^4}{x_0^4}; \frac{1}{x_0}; \frac{y_0^8}{x_0^7}; \frac{y_0^2}{x_0^2}; \frac{y_0^5}{x_0^4}; x_0; \frac{y_0}{x_0}; \frac{x_0^2}{y_0^2}; \frac{y_0^4}{x_0^4}; \frac{1}{x_0}; \frac{y_0^8}{x_0^7}; \frac{y_0^2}{x_0^2}; \frac{y_0^5}{x_0^4} x_0; \ldots\right\},$$

$$y(n) = \left\{\frac{x_0}{y_0}; \frac{y_0^2}{x_0^2}; \frac{x_0^4}{y_0^4}; \frac{y_0^8}{x_0^8}; \frac{y_0}{x_0}; \frac{y_0^7}{x_0^6}; \frac{y_0^4}{x_0^4}; y_0; \frac{x_0}{y_0}; \frac{y_0^2}{x_0^2}; \frac{x_0^4}{y_0^4}; \frac{y_0^8}{x_0^8}; \frac{y_0}{x_0}; \frac{y_0^7}{x_0^6}; \frac{y_0^4}{x_0^4}; y_0; \ldots\right\}.$$

Theorem 3: Let (x_n, y_n) be a solution of equation (1) for

$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_0 < y_{-1} < y_{-2}, \; 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_0 < y_{-2} < y_{-1},$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_0 < y_{-1} < y_{-3}, \; 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_0 < y_{-3} < y_{-1},$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_0 < y_{-2} < y_{-3}, \; 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_0 < y_{-3} < y_{-2}.$$

Then for $n = 0,1,\ldots$ we have:

$$x(n) = \left\{\frac{y_0}{x_0}; \frac{x_0^2}{y_0^2}; \frac{y_0^4}{x_0^4}; \frac{x_0^8}{y_0^8}; \frac{y_0^{16}}{x_0^{16}}; \frac{y_0^2}{x_0^2}; \frac{y_0^{13}}{x_0^{13}}; \frac{y_0^8}{x_0^8}; \frac{y_0}{x_0}; \frac{x_0^2}{y_0^2}; \frac{y_0^4}{x_0^4}; \frac{x_0^8}{y_0^8}; \frac{y_0^{16}}{x_0^{16}}; \frac{y_0^2}{x_0^2}; \frac{y_0^{13}}{x_0^{13}}; \frac{y_0^8}{x_0^8}; \ldots\right\},$$

$$y(n) = \left\{\frac{x_0}{y_0}; \frac{y_0^2}{x_0^2}; \frac{x_0^4}{y_0^4}; \frac{y_0^8}{x_0^8}; \frac{y_0^{15}}{x_0^{15}}; \frac{y_0^4}{x_0^4}; \frac{y_0^4}{x_0^4}; \frac{y_0^9}{x_0^9}; \frac{x_0}{y_0}; \frac{y_0^2}{x_0^2}; \frac{x_0^4}{y_0^4}; \frac{y_0^8}{x_0^8}; \frac{y_0^{15}}{x_0^{15}}; \frac{y_0^4}{x_0^4}; \frac{y_0^4}{x_0^4}; \frac{y_0^9}{x_0^9}; \ldots\right\}.$$

Theorem 4: Let (x_n, y_n) be a solution of equation (1) for

$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-2} < y_0 < y_{-3}\ ,\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-3} < y_0 < y_{-2}\,.$$

Then for $n = 0,1,\ldots$ we have:

$$x(n) = \left\{ \frac{y_0}{x_0}; \frac{x_0^{\ 2}}{y_0^{\ 2}}; \frac{y_0^{\ 4}}{x_0^{\ 4}}; \frac{1}{x_0}; \frac{y_0^{\ 8}}{x_0^{\ 7}}; \frac{y_0^{\ 2}}{x_0^{\ 2}}; \frac{y_0^{\ 5}}{x_0^{\ 4}}; x_0; \frac{y_0}{x_0}; \frac{x_0^{\ 2}}{y_0^{\ 2}}; \frac{y_0^{\ 4}}{x_0^{\ 4}}; \frac{1}{x_0}; \frac{y_0^{\ 8}}{x_0^{\ 7}}; \frac{y_0^{\ 2}}{x_0^{\ 2}}; \frac{y_0^{\ 5}}{x_0^{\ 4}} x_0; \ldots \right\},$$

$$y(n) = \left\{ \frac{x_0}{y_0}; \frac{y_0^{\ 2}}{x_0^{\ 2}}; \frac{x_0^{\ 4}}{y_0^{\ 4}}; \frac{y_0^{\ 8}}{x_0^{\ 8}}; \frac{y_0}{x_0}; \frac{y_0^{\ 7}}{x_0^{\ 6}}; \frac{y_0^{\ 4}}{x_0^{\ 4}}; y_0; \frac{x_0}{y_0}; \frac{y_0^{\ 2}}{x_0^{\ 2}}; \frac{x_0^{\ 4}}{y_0^{\ 4}}; \frac{y_0^{\ 8}}{x_0^{\ 8}}; \frac{y_0}{x_0}; \frac{y_0^{\ 7}}{x_0^{\ 6}}; \frac{y_0^{\ 4}}{x_0^{\ 4}}; y_0; \ldots \right\}.$$

3. Examples

Example 1: If the initial conditions are selected in accordance with Lemma 1 and Theorem 1;

$$x_{-3} = 3;\ x_{-2} = 7;\ x_{-1} = 8;\ x_0 = 10;\ y_{-3} = 13;\ y_{-2} = 17;\ y_{-1} = 20;\ y_0 = 12,$$

$$x_n = \{1.2, 0.694444, 2.0736,\ 0.232568, 18.4884, 1.44, 10.6993,\ 4.29982,$$
$$1.2, 0.694444, 2.0736,\ 0.232568, 18.4884, 1.44, 10.6993,\ 4.29982, \ldots\},$$

$$y_n = \{0.833333,\ 1.44,\ 0.482253,\ 4.29982,\ 1.2,\ 15.407,\ 2.0736,\ 5.15978,$$
$$0.833333,\ 1.44,\ 0.482253,\ 4.29982,\ 1.2,\ 15.407,\ 2.0736,\ 5.15978, \ldots\},$$

solutions are obtained and the graphs of the solutions are shown below.

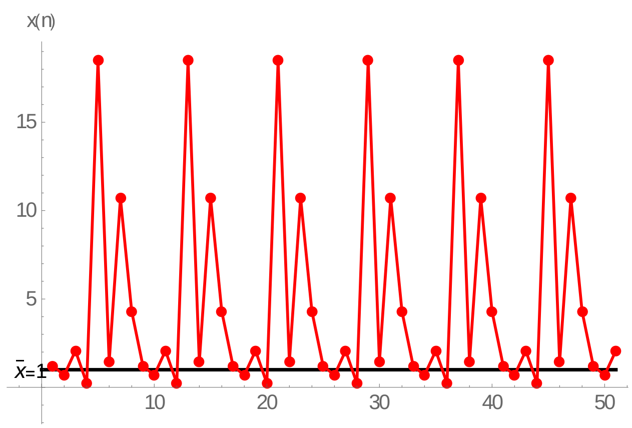

Figure 1. x_n *solutions graph.*

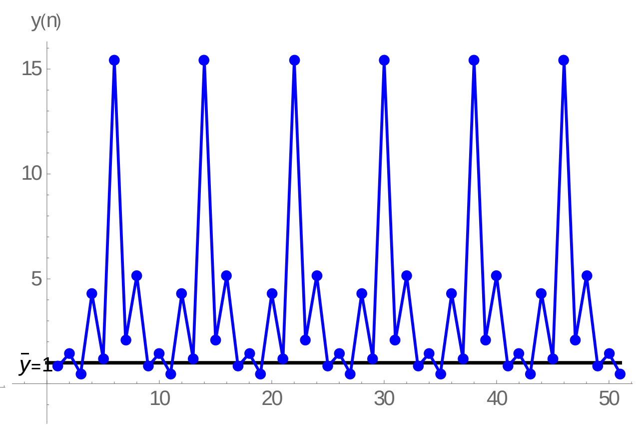

Figure 2 y_n *solutions graph.*

Example 2: If the initial conditions are selected in accordance with Lemma 2 and Theorem 2;

$$x_{-3} = 12;\ x_{-2} = 13;\ x_{-1} = 14;\ x_0 = 15;\ y_{-3} = 16;\ y_{-2} = 17;\ y_{-1} = 18;\ y_0 = 19,$$

$$x_n = \{1.26667,\ 0.623269,\ 2.57424,\ 0.150904,\ 43.9134,\ 1.60444,\ 21.6078,\ 6.62672,$$
$$1.26667,\ 0.623269,\ 2.57424,\ 0.150904,\ 43.9134,\ 1.60444,\ 21.6078,\ 6.62672,\ \ldots\},$$

$$y_n = \{0.789474,\ 1.60444,\ 0.388464,\ 6.62672,\ 1.26667,\ 34.6685,\ 2.57424,\ 8.39385,$$
$$0.789474,\ 1.60444,\ 0.388464,\ 6.62672,\ 1.26667,\ 34.6685,\ 2.57424,\ 8.39385,\ \ldots\},$$

solutions are obtained and the graphs of the solutions are shown below.

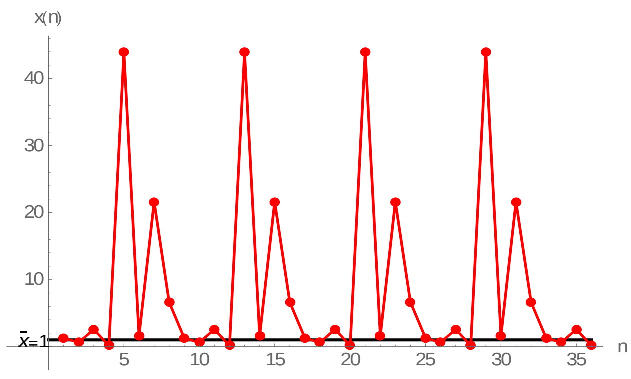

Figure 3. x_n *solutions graph.*

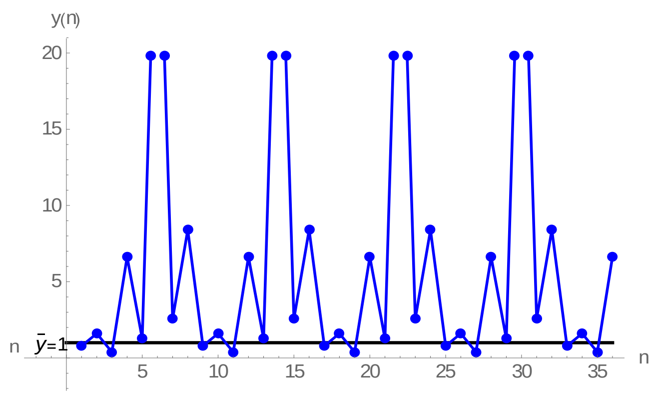

Figure 4. y_n *solutions graph.*

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